

Supersymmetry Constraints in Holographic Gravities

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Abstract

Supersymmetric higher derivative gravities define superconformal field theories via the AdS/CFT correspondence. From the boundary theory viewpoint, supersymmetry implies a relation between the coefficients which determine the three point function of the stress energy tensor which can be tested in the dual gravitational theory. We use this relation to formulate a necessary condition for the supersymmetrization of higher derivative gravitational terms. We then show that terms quadratic in the Riemann tensor do not present obstruction to supersymmetrization, while generic higher order terms do. For technical reasons, we restrict the discussion to seven dimensions where the obstruction to supersymmetrization can be formulated in terms of the coefficients of Weyl anomaly, which can be computed holographically.

Higher derivative terms in gravity naturally arise in the low energy limit of string theories. It is interesting to investigate the role these terms play in the AdS/CFT correspondence [1-3]. Common lore states that these terms encode corrections due to deviation from the large N or infinite coupling limit in the boundary theory. In fact, higher derivative terms are necessary if one wants to study the holographic duals of four-dimensional CFTs with different a and c central charges of the conformal algebra.

Once these higher derivative terms are introduced, a natural question is whether the gravity theory can be made supersymmetric. An affirmative answer presumably implies that the dual boundary theory is superconformal. The two and three point functions of the stress energy tensor of conformal field theories in $d > 3$ dimensions are completely specified by three coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}$. The superconformal Ward identity further reduces the number of independent coefficients to two; the explicit form of the constraint in four dimensions has been worked out in [4]. In six dimensions the form of the constraint has been determined in [5], and it is not hard to generalize this to arbitrary dimensions. The three point functions of the stress energy tensor are related to graviton scattering amplitudes in AdS. (These on-shell amplitudes, and consequently, the results of this paper, are unaffected by the field redefinitions in the bulk.) One should therefore be able to test whether there is an obstacle to making a given gravity theory supersymmetric by computing these scattering amplitudes and checking whether the constraint is satisfied¹.

For technical reasons, we consider gravities in seven dimensions which are dual to the six dimensional CFTs. This is because the latter have a peculiar property; the three coefficients in front of the B-type terms in the Weyl anomaly, which we denote by $b_n, n = 1, \dots, 3$, are linearly related to $\mathcal{A}, \mathcal{B}, \mathcal{C}$. Hence, supersymmetry implies a linear relation between b_n . More precisely, consider the Weyl anomaly in the form

$$\mathcal{A}_W = E_6 + \sum_n^3 b_n I_n + \nabla_i J^i \quad (1)$$

where E_6 is the Euler density in six dimensions, $I_n, n = 1, \dots, 3$ are three independent conformal invariants composed out of the Weyl tensor and its derivatives, and the last

¹ It would be interesting to generalize this to include other fields, but we leave this for future work.

term is a total derivative of a covariant expression. The free field theory result for the B-type part of the Weyl anomaly is [6]

$$\begin{aligned} b_1 &= \frac{28}{3}n_s + \frac{896}{3}n_f + \frac{8008}{3}n_a \\ b_2 &= \frac{5}{3}n_s - 32n_f - \frac{2378}{3}n_a \\ b_3 &= 2n_s + 40n_f + 180n_a \end{aligned} \tag{2}$$

where n_s, n_a, n_f are the numbers of scalar fields, antisymmetric two-forms and Dirac fermions in six dimensions. In terms of free fields, the supersymmetry condition can be written as

$$6n_a + n_s - 8n_f = 0 \tag{3}$$

This defines a plane in the (n_a, n_s, n_f) space which passes through the origin, $(n_a = 0, n_s = 8, n_f = 1)$ and the point $(n_a = 1, n_s = 10, n_f = 2)$. The former corresponds to a free scalar superfield in 6 dimensions while the latter to two (2,0) multiplets. Since the relations between (n_a, n_s, n_f) , $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and (b_1, b_2, b_3) are linear, eq. (3) allows one to determine the explicit form of the constraint that supersymmetry imposes on b_n . The result is

$$b_1 - 2b_2 + 6b_3 = 0 \tag{4}$$

In the following we are going to check if this constraint is satisfied to leading order in the coefficients in front of the higher derivative terms in the gravitational lagrangian.

Consider the following action with negative cosmological constant:

$$S = \int \sqrt{-g} \mathcal{L} = \int \sqrt{-g} \left(R + \frac{30}{L^2} + \sum_i \mathcal{L}_i \right), \quad \mathcal{L}_i = \sum_j a_{ij} \mathcal{L}_{ij} \tag{5}$$

In (5) \mathcal{L}_i stand for all possible $\mathcal{O}(R^i)$ higher derivative terms, while \mathcal{L}_{ij} denotes all possible contractions of the Riemann tensors which are $\mathcal{O}(R^i)$. An AdS space of length $L + \mathcal{O}(a_{ij})$ is a solution of the equations of motion. The explicit expressions for the higher derivative terms \mathcal{L}_2 and \mathcal{L}_3 are

$$\begin{aligned} \mathcal{L}_2 &= a_{21} R_{MNPQ}^2 + a_{22} R_{MN}^2 + a_{23} R^2 \\ \mathcal{L}_3 &= a_{31} R^{IJKL} R_{KLMN} R_{IJ}^{MN} + a_{32} R^{IJ}{}_{KM} R^{KL}{}_{JN} R_{IL}^{MN} + a_{33} R^{IJKL} R_{KLJM} R^M{}_I + \\ &\quad + a_{34} R R_{IJKL}^2 + a_{35} R^{IKJL} R_{JI} R_{LK} + a_{36} R^{IJ} R_{JK} R^K{}_I + a_{37} R R_{IJ}^2 + a_{38} R^3 \end{aligned} \tag{6}$$

We will also consider an $\mathcal{O}(R^4)$ term of the type

$$\mathcal{L}_4 = a_{41} R^{IJKL} R_{KLMN} R^{MNPQ} R_{PQIJ} + \dots \quad (7)$$

In the following we will compute the leading corrections to the b_n from all terms in (6), (7). The leading (Einstein-Hilbert) result, $b_1^{(0)} = -1680$, $b_2^{(0)} = -420$, $b_3^{(0)} = 140$ [7-8] satisfies (4). Each term in (6), (7) is going to give rise to

$$b_n = b_n^{(0)} + a_{ij} \tilde{b}_n^{(ij)} + \mathcal{O}(a_{ij}^2), \quad n = 1, \dots, 3 \quad (8)$$

Let us introduce

$$B^{(ij)} = \tilde{b}_1^{(ij)} - 2\tilde{b}_2^{(ij)} + 6\tilde{b}_3^{(ij)} \quad (9)$$

The condition

$$\sum_{ij} a_{ij} B^{(ij)} \neq 0 \quad (10)$$

implies that there is an obstruction to the supersymmetrization of the corresponding term in higher derivative gravity.

To compute $\tilde{b}^{(ij)}$ we will make use of the prescription developed in [7-8]. In practice, we will mostly follow [5]. Consider the Einstein-Hilbert action with negative cosmological constant, $a_{ij} = 0$, and the following ansatz for the metric

$$ds^2 = L^2 \left(\frac{1}{4\rho^2} d\rho^2 + \frac{1}{\rho} g_{\mu\nu} dx^\mu dx^\nu \right) \quad (11)$$

where

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \rho g_{\mu\nu}^{(1)} + \rho^2 g_{\mu\nu}^{(2)} + \rho^3 g_{\mu\nu}^{(3)} + \mathcal{O}(\rho^3 \log \rho) \quad (12)$$

is an expansion in powers of the radial coordinate ρ . One can now solve the Einstein equations of motion order by order in the ρ expansion and determine $g_{\mu\nu}^{(p)}$, $p = 1, \dots$ in terms of $g_{\mu\nu}^{(0)}$. The resulting expansion (12) is then substituted back into the Einstein-Hilbert action density and the coefficient of the $1/\rho$ term encodes the anomaly in Einstein-Hilbert gravity. To compute the $\mathcal{O}(a_{ij})$ correction to the anomaly it is sufficient to evaluate the $\sqrt{\det g} \mathcal{L}_{ij}$ term on the solutions of the Einstein equations of motion and extract the $1/\rho$ term. This is because the $\mathcal{O}(a_{ij})$ contribution from the Einstein-Hilbert lagrangian due

to the $\mathcal{O}(a_{ij})$ change in the solution (12) is proportional to the equations of motion and vanishes on-shell². In other words, we are going to compute

$$\sqrt{\det g^{(0)}} \mathcal{A}_W^{(ij)} = \left[\sqrt{\det g} \mathcal{L}_{ij}(g_{ij}) \right]_{\frac{1}{\rho}} = \left[\sqrt{\det g^{(0)}} \mathcal{A}_W^{(ij)} \right]_{g^{(0,1,2)}} + \left[\sqrt{\det g^{(0)}} \mathcal{A}_W^{(ij)} \right]_{g^{(3)}} \quad (13)$$

where $[\dots]_{\frac{1}{\rho}}$ means that we are extracting $1/\rho$ coefficient from the expression in the square brackets. In (13) \mathcal{L}_{ij} is evaluated on the solution (12) of Einstein equations of motion; for technical reasons it is convenient to separate the contributions from $g^{(0,1,2)}$ and $g^{(3)}$. In particular, the former piece

$$\left[\sqrt{\det g^{(0)}} \mathcal{A}_W^{(ij)} \right]_{g^{(0,1,2)}} = \left[\sqrt{\det g} \mathcal{L}_{ij} \left(g = g^{(0)} + \rho g^{(1)} + \rho^2 g^{(2)} \right) \right]_{\frac{1}{\rho}} \quad (14)$$

is evaluated on the metric truncated to the $\mathcal{O}(\rho^2)$. As in [5] we take the boundary metric to be of the form

$$g_{\mu\nu} dx^\mu dx^\nu = f(x^3, x^4) [(dx^1)^2 + (dx^2)^2] + \sum_{i=3}^6 (dx^i)^2 \quad (15)$$

and use Mathematica to determine $g^{(1)}$,

$$g_{\mu\nu}^{(1)} = -\frac{1}{4} \left(R_{\mu\nu} - \frac{1}{10} R g_{\mu\nu}^{(0)} \right) \quad (16)$$

and $g^{(2)}$ (which is slightly more complicated, so we do not quote it here). In eq. (16) $R_{\mu\nu}$ is the curvature tensor of the metric $g^{(0)}$. This way we completely determine (14).

Unlike [5], we also need an expression for $g^{(3)}$ which contributes to the $\mathcal{O}(a_{ij})$ term in the anomaly. This is because we used the equations of motion to eliminate the correction coming from the Einstein-Hilbert lagrangian. It is easier to find $g^{(1)}$ and $g^{(2)}$ rather than $g^{(3)}$ because the $\mathcal{O}(\rho^3 \log \rho)$ in (12) contributes to the equations of motion at this order. Fortunately, the contribution to \tilde{b}_{ij} due to $g^{(3)}$ comes in a very simple form. Namely, the term in the anomaly (13) due to $g^{(3)}$ is given by

$$\left[\mathcal{A}_W^{(ij)} \right]_{g^{(3)}} = c_{ij} \text{Tr} \left[\left(g^{(0)} \right)^{-1} g^{(3)} \right] \quad (17)$$

² A similar approach was used in [9] in the context of $\mathcal{O}(R^2)$ corrections to Einstein gravity in AdS_5 .

where c_{ij} are easily found to be

$$c_{21} = 9, \quad c_{22} = 27, \quad c_{23} = 189 \quad (18)$$

$$c_{31} = -6, \quad c_{32} = 6, \quad c_{33} = 18, \quad c_{34} = -126, \quad c_{35} = c_{36} = -54, \quad c_{37} = -378, \quad c_{38} = -2646 \quad (19)$$

and $c_{41} = -12$. To compute $\text{Tr} \left[(g^{(0)})^{-1} g^{(3)} \right]$ one can use equations of motion of Einstein-Hilbert gravity. They have been written down in a convenient form in [7]; we only need the last line of equation (7) in [7]:

$$\text{Tr} [g^{-1} g''] = \frac{1}{2} \text{Tr} [g^{-1} g' g^{-1} g'] \quad (20)$$

Here g is the metric (12) and prime denotes differentiation with respect to ρ . Substituting the expansion (12) into (20) one can use the $\mathcal{O}(\rho)$ term in the resulting expression to write

$$\begin{aligned} \text{Tr} \left[(g^{(0)})^{-1} g^{(3)} \right] &= \frac{1}{6} \left(4 \text{Tr} \left[(g^{(0)})^{-1} g^{(1)} (g^{(0)})^{-1} g^{(1)} \right] \right. \\ &\quad \left. - \text{Tr} \left[(g^{(0)})^{-1} g^{(1)} (g^{(0)})^{-1} g^{(1)} (g^{(0)})^{-1} g^{(1)} \right] \right) \end{aligned} \quad (21)$$

Together with (18), (19) and the solution for $g^{(1)}, g^{(2)}$, eq. (21) allows us to compute (17). Combining this with (14), we obtain an expression for the Weyl anomaly. We then demand that the coefficient in front of every term in the expression

$$\mathcal{A}_W^{(ij)} - \sum_{n=1}^3 \tilde{b}_n^{(ij)} I_n - \sum_{n=1}^7 c_n^{(ij)} C_n = 0 \quad (22)$$

vanishes. In eq. (22) the I_n are the B-type anomaly terms composed out of the Weyl tensor, and C_n are the total derivative terms. Both can be found in Appendix A of [6]. This completely fixes \tilde{b}_n and c_n . The results are summarized below (we omit an overall coefficient common to all $\tilde{b}_n^{(ij)}$).

$$\begin{aligned} \tilde{b}_1^{(21)} &= \frac{5}{96}, & \tilde{b}_2^{(21)} &= \frac{37}{384}, & \tilde{b}_3^{(21)} &= \frac{3}{128}; & B^{(21)} &= 0 \\ \tilde{b}_1^{(22)} &= \frac{21}{32}, & \tilde{b}_2^{(22)} &= \frac{21}{128}, & \tilde{b}_3^{(22)} &= -\frac{7}{128}; & B^{(22)} &= 0 \\ \tilde{b}_1^{(23)} &= \frac{147}{32}, & \tilde{b}_2^{(23)} &= \frac{147}{128}, & \tilde{b}_3^{(23)} &= -\frac{49}{128}; & B^{(23)} &= 0 \end{aligned} \quad (23)$$

Thus all $\mathcal{O}(R^2)$ terms have vanishing B . The results for the $\mathcal{O}(R^3)$ terms are

$$\begin{aligned}
\tilde{b}_1^{(31)} &= \frac{9}{16}, & \tilde{b}_2^{(31)} &= \frac{9}{64}, & \tilde{b}_3^{(31)} &= -\frac{41}{192}; & B^{(31)} &= -1 \\
\tilde{b}_1^{(32)} &= \frac{23}{16}, & \tilde{b}_2^{(32)} &= \frac{7}{64}, & \tilde{b}_3^{(32)} &= -\frac{31}{192}; & B^{(32)} &= \frac{1}{4} \\
\tilde{b}_1^{(33)} &= \frac{5}{16}, & \tilde{b}_2^{(33)} &= \frac{37}{64}, & \tilde{b}_3^{(33)} &= \frac{9}{64}; & B^{(33)} &= 0 \\
\tilde{b}_1^{(34)} &= -\frac{35}{16}, & \tilde{b}_2^{(34)} &= -\frac{259}{64}, & \tilde{b}_3^{(34)} &= -\frac{63}{64}; & B^{(34)} &= 0 \\
\tilde{b}_1^{(35)} &= -\frac{63}{16}, & \tilde{b}_2^{(35)} &= -\frac{63}{64}, & \tilde{b}_3^{(35)} &= -\frac{21}{64}; & B^{(35)} &= 0; & \tilde{b}_n^{(35)} &= \tilde{b}_n^{(36)} \\
\tilde{b}_1^{(37)} &= -\frac{441}{16}, & \tilde{b}_2^{(37)} &= -\frac{441}{16}, & \tilde{b}_3^{(37)} &= \frac{147}{64}; & B^{(37)} &= 0; \\
\tilde{b}_1^{(38)} &= -\frac{3087}{16}, & \tilde{b}_2^{(38)} &= -\frac{3087}{16}, & \tilde{b}_3^{(38)} &= \frac{1029}{64}; & B^{(38)} &= 0;
\end{aligned} \tag{24}$$

Finally,

$$\tilde{b}_1^{(41)} = -\frac{25}{8}, \quad \tilde{b}_2^{(41)} = -\frac{89}{32}, \quad \tilde{b}_3^{(41)} = \frac{89}{96}; \quad B^{(41)} = 8 \tag{25}$$

Let us discuss these results. Apparently, there is no obstruction to supersymmetrizing $\mathcal{O}(R^2)$ terms, at least at the linear level. This is consistent with the results of [5] where the supersymmetric constraint for the Gauss-Bonnet term has been shown to hold. It seems that this statement might be dimension independent (the analogous quantity has been shown to vanish in any dimensions [10]; see also [11]).

The situation with cubic terms is more interesting. Apparently a generic term cubic in the Riemann tensor cannot be supersymmetrized. This is consistent with the fact that such terms do not appear in superstring amplitudes [12]. Note that it is possible to take a linear combination of the $\mathcal{O}(R^3)$ terms to engineer $B = 0$. In fact, according to the recent results of [13] the cubic Lovelock term (Euler density in six dimensions) is precisely of this type which implies that there is no obstruction to supersymmetrizing this term. Note that the Lovelock term vanishes upon dimensional reduction to four dimensions, which is consistent with the expectations that $\mathcal{O}(R^3)$ terms cannot be supersymmetrized there³. It is also interesting to observe that the generic $\mathcal{O}(R^4)$ term contributes to the anomaly, and hence to the three point function of the stress energy tensor. This contribution leads to a non-vanishing value of B .

To summarize, we formulated the necessary condition for supersymmetry in CFTs dual to higher derivative gravities. Non-vanishing B defined by (9) implies that the boundary

³ We thank Martin Rocek for explaining this point to us.

theory is not superconformal. This can be used to check whether the relevant term in higher derivative gravity can be supersymmetrized or not. In particular, we found that $B = 0$ for all terms of $\mathcal{O}(R^2)$ but is generically nonvanishing for the $\mathcal{O}(R^3)$ and $\mathcal{O}(R^4)$ terms. We did not investigate $\mathcal{O}(R^5)$ and higher derivative terms but see no reason why they would generically lead to vanishing B .

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